قضیه صفحه ۶

***Theorem 4.*** *Every* m $\times$ n *matrix over the field* F *is row-equivalent to a row-reduced matrix.*

*Proof.* Let $A$ be an $m \times n$ matrix over $F$. If every entry in the first row of $A$ is 0, then condition (a) is satisfied in so far as row 1 is concerned. If row 1 has a non-zero entry, let $k$ be the smallest positive integer $j$ for which $A_{1j} \neq 0$. Multiply row 1 by $A_{1k}^{-1}$, and then condition (a) is satisfied with regard to row 1. Now for each $i \geq 2$, add $(-A_{ik})$ times row 1 to row $i$. Now the leading non-zero entry of row 1 occurs in column $k$, that entry is 1, and every other entry in column $k$ is 0.

Now consider the matrix which has resulted from above. If every entry in row 2 is 0, we do nothing to row 2. If some entry in row 2 is different from 0, we multiply row 2 by a scalar so that the leading non-zero entry is 1. In the event that row 1 had a leading non-zero entry in column $k$, this leading non-zero entry of row 2 cannot occur in column $k$; say it occurs in column $k_r \neq k$. By adding suitable multiples of row 2 to the various rows, we can arrange that all entries in column $k'$ are 0, except the 1 in row 2. The important thing to notice is this: In carrying out these last operations, we will not change the entries of row 1 in columns $1, \ldots, k$; nor will we change any entry of column $k$. Of course, if row 1 was identically 0, the operations with row 2 will not affect row 1.

Working with one row at a time in the above manner, it is clear that in a finite number of steps we will arrive at a row-reduced matrix. ∎

## قضیه صفحه ۱۱

***Theorem 5.*** *Every* m × n *matrix* A *is row-equivalent to a row-reduced echelon matrix.*

*Proof.* We know that $A$ is row-equivalent to a row-reduced matrix. All that we need observe is that by performing a finite number of row interchanges on a row-reduced matrix we can bring it to row-reduced echelon form. ∎

## قضایای اسلاید ۱۵

**Theorem 3.10.1.** *For every matrix $A$, there is a sequence of row operations taking $A$ to a matrix in row reduced echelon form.*

*Proof.* We prove this by induction on the number of columns of $A$. When $A$ has one column, either $A$ is the zero vector (in which case it is already in RREF) or it has a nonzero entry $a$. Swap $a$ to the top row, multiply the top row by $1/a$, and use the $1, 1$ entry as a pivot to eliminate the other entries of $A$. The result is the vector with a 1 at the top and zeroes elsewhere, which is in RREF.

For the inductive step, suppose that $A$ is $m \times n$ and that the result is true for all matrices with $n-1$ columns. We then know that there is a series of row operations we can do to $A$ that result in a matrix $X$ whose first $n-1$ columns form a RREF matrix. Suppose the matrix formed by these $n-1$ columns has $k$ rows of zeroes at the bottom. If the final column has zeroes in its bottom $k$ entries, the matrix is in RREF. If not, swap a nonzero entry to the top of these $k$ rows, use it as a pivot to eliminate all other nonzero entries in the final column, and multiply by a scalar so that its entry is 1. The result is in RREF. ∎

**Theorem 3.10.2.** *Let $A$ be a matrix. If $R$ and $S$ are RREF matrices that can be obtained by doing row operations to $A$, then $R = S$.*

This theorem says that there is only one RREF matrix which can be obtained by doing row operations to $A$, so we are justified in calling the unique RREF matrix reachable from $A$ *the* row reduced echelon form of $A$.

*Proof.* Again, the proof is by induction on the number $n$ of columns of $A$. There are only two RREF column vectors: the zero vector and a vector with a 1 at the top and all other entries zero. Clearly no sequence of row operations takes one of these to the other, so the base case of the induction holds.

For the inductive step, suppose that $R$ and $S$ are RREF matrices reachable from $A$. Let $A'$, $R'$, and $S'$ be the matrices formed by the first $n-1$ columns of $A$, $R$, and $S$ respectively. The matrices $R'$ and $S'$ are RREF matrices formed by doing row operations to $A'$, so by induction they are equal. Suppose for a contradiction that $R \neq S$, so that there is some $j$ such that the $j$th entry $r_{jn}$ in the last column of $R$ which differs from the corresponding entry $s_{jn}$ of $S$.

## قضیه اسلاید ۲۳

برای این اثبات از قضیه‌ای جلسه قبل اثبات کردید و در زیر صفحه مجدد آوردیم استفاده می‌شود به این شکل که:

***Theorem 6.*** *If* A *is an* m × n *matrix and* m < n, *then the homogeneous system of linear equations* AX = 0 *has a non-trivial solution.*

*Proof.* Let $R$ be a row-reduced echelon matrix which is row-equivalent to $A$. Then the systems $AX = 0$ and $RX = 0$ have the same solutions by Theorem 3. If $r$ is the number of non-zero rows in $R$, then certainly $r \leq m$, and since $m < n$, we have $r < n$. It follows immediately from our remarks above that $AX = 0$ has a non-trivial solution. ∎

## قضیه اسلاید ۲۴

***Theorem 7.*** *If* A *is an* n × n *(square) matrix, then* A *is row-equivalent to the* n × n *identity matrix if and only if the system of equations* AX = 0 *has only the trivial solution.*

*Proof.* If $A$ is row-equivalent to $I$, then $AX = 0$ and $IX = 0$ have the same solutions. Conversely, suppose $AX = 0$ has only the trivial solution $X = 0$. Let $R$ be an $n \times n$ row-reduced echelon matrix which is row-equivalent to $A$, and let $r$ be the number of non-zero rows of $R$. Then $RX = 0$ has no non-trivial solution. Thus $r \geq n$. But since $R$ has $n$ rows, certainly $r \leq n$, and we have $r = n$. Since this means that $R$ actually has a leading non-zero entry of 1 in each of its $n$ rows, and since these 1's occur each in a different one of the $n$ columns, $R$ must be the $n \times n$ identity matrix. ∎

اثبات صفحه ۲۹

**THEOREM 4**

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$.

d. $A$ has a pivot position in every row.



Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$. The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

***Warning:*** Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.